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Point-wise Regularity Exponents and Markets Cross-Correlation

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The multifractional Brownian motion is a locally dependent Gaussian nonstationary process, whose flexibility in describing complex phenomena justifies its use in financial dynamics modeling. Assuming it as a model of stock indexes, we estimate the pointwise regularity function for the Dow Jones Ind. Avg., the Footsie 100 and the Nikkei 225. We also analyze the pairwise cross-correlation of the functions themselves and compare them with the pairwise cross-correlation of log variations.

Field of Research: multifractional Brownian motion, pointwise regularity, cross-correlation.

1. Introduction

A large amount of contributions, mostly empirical, have provided in the last years significant evidence that seriously weakened the paradigm of the standard financial theory: Brownian motion and its derivations. A renewed emphasis on this topic has been added by the recent global crisis and the consequent crashes in international financial markets: shocks that according to theory should have negligible probability to occur in point of fact come true with a frequency of orders of magnitude larger than those foreseen by the model. The challenge is therefore building models able to preserve the elegant no-arbitrage theory of the Brownian world and to encompass at the same pace the turbulence displayed by the empirical dynamics. In the last thirty years a number of attempts have been addressed towards this direction, generally relaxing the hypothesis of independence or the assumption of identical distribution of the price variations.

In this paper we take into consideration a very challenging process that relaxes both these assumptions: the multifractional Brownian motion (mBm), a nonstationary dependent Gaussian process, that generalizes the fractional Brownian motion (fBm) replacing the parameter H by a Hölderian function of time (see Figure 1). This accounts for the pointwise regularity of the process at each time t (values of H_t larger than ½ indicate persistence, the stronger the higher they are; on the contrary, values of H_t smaller than ½ denote mean-reversion, the stronger the lower they are). Although the difficulty to infer global probabilistic properties for this model has curbed its spread in finance, there are several reasons suggesting that the mBm deserves more attention. We quote the following:

(a) choosing in a proper and realistic way its functional parameter, the mBm is fit to reproduce the stylized facts empirically observed in financial time series (Cont, 2001): absence of autocorrelation in returns and significantly autocorrelated absolute/square

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returns, unconditional and conditional heavy tails, gain/loss asymmetry, intermittency, aggregational Gaussianity, volatility clustering and volume/volatility correlation;

(b) Bianchi and Pianese (2008) pointed out that the time varying dependence characterizing the mBm can harmonize the many inconsistent estimates that literature provides of the long-range dependence parameter. In fact, it can be easily shown that detecting dependence or not using asymptotic estimators strongly relies on the segment of data one looks at, even for the same time series, what indicates nonstationarity (Bianchi, 2005b). As recalled in the next section, the presence of dependence in financial time series is a key issue for its implications in pricing but also with regard to the reliability of the cross-correlation estimation;

(c) Most importantly, the mBm succeeds in giving a rationale to the market mechanism (Bianchi, 2005a; Frezza, 2008). In fact, the estimates of the function H_t (the very core of the mBm) are centered around ½, the sole value consistent with the no-arbitrage condition within this class of models. Moreover, the paths of H_t show sudden downturns as much large as much severe are the crises hitting the market; the falls are generally followed by gradual upturns that tend to restore the "regularity level" preceding the falls themselves. This behaviour captures the asymmetric impact of the information flow; since a trend is the result of a gradual increase of confidence and a crash is generally a sudden event, the former corresponds to values of H_t larger than ½, whereas the latter coincides with low values of H_t. These seize the quick buy-and-sell activity, which means mean-reversion, typical of the crisis periods.

The mechanism described in (c) links in a very parsimonious way the trading style with the latent explanatory variable H_t; under the assumption of multifractionality, this function provides relevant information about the pointwise state of the market, i.e. about how the market itself perceives and weighs the past information. In this paper, after having estimated the values of H_t for three main stock indexes in the period 1984-2010, we use the estimates to analyze the pairwise cross-correlations of the three markets in terms of pointwise regularity rather than of returns. The aim is to look for an innermost cross-correlation that returns are not able to catch. The paper is organized as follows: in Section 2 we summarize the state of the art about the estimation of the poitwise Hölder exponent; in Section 3 the moving average integral representation of the mBm and the class of estimators are introduced. We also provide the optimal settings of the parameters, in terms of reduction of the estimator's variance, and we discuss the financial intuition for H_t. In Section 4 is devoted to the empirical application: we analyze the pairwise cross-correlation of the pointwise regularity functions of three main indexes and compare them with the pairwise cross-correlation of log variations. Conclusion and further development are discussed in the last section.

2. Literature Review

Concerning the financial modelling, many contributions in the last years have proposed estimators of the mBm's functional parameter H_t, mainly based on two approaches: on the one hand the average variation of a Gaussian random variable and on the other hand the generalized quadratic variation. The former group of contributions started with the work of Péltier and Lévy Véhel (1994), who proposed an estimator based on the absolute moment of a normal random variable in order to evaluate the parameter H of an fBm. Their results were extended to cover the case of the functional parameter of an mBm by Bianchi (2005a), with a particular view to financial datasets. The latter field of investigation was opened by the works of Istas and Lang (1997) and Benassi et al. (1998); both defined an estimator, based on generalized quadratic variations, of a continuously differentiable function allowing Gaussian limiting distribution. Coeurjolly (2005) extended the results from the continuously differentiable functions to the Hölderian functions of arbitrary positive order. Further developments were achieved in order to detect abrupt changes of the Hölder exponent of Gaussian processes with almost sure continuous paths: a semi-parametric estimator for a piece-wise constant H_t was proposed by Benassi et al. (1999) and by Benassi et al. (2000). Ayache and Lévy Véhel (2004), Ayache et al. (2005) and Ayache et al. (2007) used the generalized quadratic variation and derived a central limit theorem for the estimator of the multifractional function of a process even more general than the mBm (what they called the Generalized multifractional Brownian motion, GmBm). In authors' knowledge, the most recent contribution – which seems to open a third approach in the estimation of the multifractional functional – is due to Coeurjolly (2008), who introduced a class of consistent estimators based on convex combinations of sample quantiles of a sample path's discrete variations and derived the almost sure convergence and the asymptotic normality of the estimators.

Concerning the cross-correlations, many contributions documented that they are higher when markets are more volatile. Although this seems to hold considering both the unconditional correlations and the time-varying conditional ones, the issue is somewhat controversy. For example, Ramchand and Susmel (1998) found that the correlations between the U.S. and other world markets are on average 2 to 3.5 times higher (with respect to low volatile periods) when the U.S. market experiences high volatility; Forbes and Rigobon (2002) showed that correlation coefficients are conditional on market volatility, so they are biased measures of dependence when markets become more volatile. Once the series are adjusted for this bias, no evidence of contagion arises even in presence of financial crises such as the 1987 U.S. market crash, the 1994 Mexican devaluation or the 1997 Asian crisis. Bartram and Wang (2004) stated that a bias in dependence measures occurs only for particular assumptions about the time-series dynamics and argued that, being real world data often characterized by heteroskedasticity, a correction of the estimated unconditional correlations during market crises may not always be needed. Differently, if the return process displays conditional heteroskedasticity, a bias exists that cannot be distinguished from a fundamental change in market dependence. Knif et al. (2005) found weaker evidence of correlations driven by market downturns and provide evidence that correlations have been increasing between national markets in recent years.

3. Identification of the mBm's pointwise regularity

3.1 The mBm

In this subsection we will sketch the main properties of the multifractional Brownian motion in order to stress only those issues of interest for the application that will be discussed into the next section. Before, it is worthwhile recalling that this model – in spite of the assonance – should not be confused with the multifractal one, which belongs to the family of the time-changed processes.

Broadly speaking, the mBm (that will be denoted by $W_{H_t}(t)$) can be obtained through two independent ways, one based on the moving average integral representation (Péltier and

Lévy Véhel, 1995) and the other based on a spectral approach (Benassi et al., 1998). We will refer to the former representation, that reads as

$$W_{H_t}(t) = \frac{1}{\Gamma\left(H_t + \frac{1}{2}\right)} \left\{ \int_{-\infty}^{0} \left[(t-s)^{H_t - \frac{1}{2}} - (-s)^{H_t - \frac{1}{2}} \right] dB(s) + \int_{0}^{t} (t-s)^{H_t - \frac{1}{2}} dB(s) \right\} \tag{1}$$

that is as the weighted sum of the increments of the Brownian motion $dB(\cdot)$, with the weights depending on the Hölderian function H_t. In (1) the symbol Γ denotes the function gamma and t is the time the process is evaluated at. The process has covariance given by

$$E\left(W_{H_t}(t) W_{H_s}(s)\right) = D(H_t, H_s)\left(t^{H_t+H_s} + s^{H_t+H_s} - |t-s|^{H_t+H_s}\right)$$

where the quantity

$$D(H_t, H_s) = \frac{\left(\Gamma(2H_t+1)\Gamma(2H_s+1)\sin(\pi H_t)\sin(\pi H_t)\right)^{1/2}}{2\Gamma(H_t+H_s+1)\sin\left(\pi \frac{H_t+H_s}{2}\right)}$$

tends to 1 as $|H_t - H_s| \to 0$. Therefore, provided that $H_t \approx H_s$, the variance of the increment process reads as

$$Var\left(W_{H_t}(t) - W_{H_s}(s)\right) \approx |t-s|^{2H_t} \tag{2}$$

A more insightful way to deduce relation (2) links the local behaviour of the mBm and the fBm. This relevant property, that can be exploited in order to estimate H_t, is the *local asymptotical self-similarity* of the mBm, in the sense stated by (Benassi et al., 1997):

$$\lim_{a \to 0^+} a^{-H_t}\left(W_{H_{t+au}}(t+au) - W_{H_t}(t)\right) \sim B_{H_t}(u), \quad u \in \mathbb{R} \tag{3}$$

where $B_{H_t}(\cdot)$ denotes the fractional Brownian motion of parameter H_t and the symbol ~ indicates the equality in distribution. Basically, equality (3) states that at any point t there exists a fBm of parameter H_t “tangent” to the mBm. The result is relevant for it suggests that the function H in a sufficiently small neighbourhood of any point t can be thought as if it were constant. Therefore one can exploit the estimators defined for the fBm to appraise the local behaviour of the function H_t, provided that the estimators themselves show a sufficiently high rate of convergence (i.e., provided that they work even for very short spans of time). This is the case of the estimator introduced by Péltier and Lévy Vèhel (1994) that will be described in the next subsection.
Finally, Figure 2 displays the autocovariance function of the increments of a $B_H(t)$. Notice that for values of H different from ½ the increments are positively (H>½) or negatively (H<½) correlated; letting H to vary through time means allowing continuous jumps from each curve to another depending on the dynamics of H_t.

3.2 Identification of H_t

Since $B_{H_t}(u)$ is normally distributed with mean 0 and variance equal to $K^2 u^{2H_t}$, where K is a scale factor, an efficient estimator of H_t for a sampled mBm $W_{t,n}$ of length n makes use of the k-th absolute moment of a zero mean Gaussian random variable

$$E\left(\left|W_{j+q,n}-W_{j,n}\right|^{k}\right)=\frac{2^{k/2}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(1/2)}K^{k}u^{kH_t}$$

and is defined as follows (Bianchi, 2005a)

$$H^{k}_{\delta,q,n,K}(t)=\frac{\log\left(\frac{\sqrt{\pi}}{\delta-q+1}\sum_{j=t-\delta}^{t-1}\left|W_{j+q,n}-W_{j,n}\right|^{k}/\left(2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^{k}\right)\right)}{k\log\left(\frac{q}{n-1}\right)} \qquad (4)$$

The class of estimators (4), written for the k^{th} absolute moment of a normal random variable sampled at n points, is referred to a moving window of length δ. The parameter q indicates the lag of the differences. The estimator, normally distributed around H_t, have a very good rate of convergence $O((\sqrt{\delta}\log n)^{-1})$ that ensures to get reliable estimates even for short samples. In the empirical analysis of Section 4, the estimates are obtained for a moving window of length equal to one trading month (25 days).

Optimal settings of the parameters, in the sense of reduction of the estimator's variance, were proved to be $k = 2$ and $q = 1$ (Bianchi (2005)). Setting erroneously the scale factor K (which is not a priori known) produces an additive shift of the estimated sequence with respect to the actual one, but Bianchi et al. (2010) correct the bias through a scaling law-based regression performed on a subsample of the estimates. In the application we will implement the estimator using the new correcting technique, whose detailed description goes beyond the aim of this work.

Figure 3 provides an example of how effective the estimator is in shadowing the function H_t for a surrogated mBm. Panel (a) shows the functional parameter H_t; notice in panel (b) how the smoothness of the stochastic process (1) increases with H_t (symmetrically, the jaggedness increases for decreasing values of H_t); finally, panel (c) shows the estimation of H_t obtained by (4) (we have superimposed the function H_t displayed in panel (a) to facilitate the evaluation of the goodness of fit).

Combining Figure 2 and Figure 3(b) makes the financial intuition of H_t clear: smooth intervals indicate positive dependence or trends (bull or bear market), that is periods in which investors give the past information a weight as much larger as the difference H_t – ½; on the contrary, jagged intervals indicate negative dependence or lateral market, that is periods in which the investors look only at more recent past information (the autocovariance function displayed in Figure 2 quickly goes to zero for H<½) and trade oppositely with respect to the market direction they observe. This behaviour is likely to take place when the market experiences crises induced by shocks that force the investors to reconsider the weight they assign to the past. So, the knowledge of H at each time for a stock index synthesizes the "sentiment" of the market.

4. Empirical application

4.1 Design of the experiment

Under the assumptions discussed in the previous sections, we estimated the function H_t and calculated the relative cross-correlation for three stock indexes: the Dow Jones Industrial Average (DJIA), the Footsie 100 (FTSE) and the Nikkei 225 (N225). The analysis covered a period of about 27 years, starting on February, 6th 1984 and ending on February, 17th 2010. As the trading calendar changes with the market, it was needed to bring into alignment the series by joining the time sets and interpolating the missing data. The procedure returned an amount of 6,763 common observations.
In order to appraise the function H_t, the pre-estimation analysis proposed in Bianchi et al. (2010) was performed. It brought to fix the parameters summarized in Table 1 for the three time series.

Table 1. Estimation parameters

	DJIA	FTSE	N225
q	1	1	1
k	2	2	2
δ	25	25	25
K	0.784	0.742	1.030

The values of $H^k_{\delta,q,n,K}(t)$ for each of the three indexes are displayed in Figures 4(a)-(c), together with the empirical distributions of the estimates fitted by the normal density.
The second step consisted in the cross-correlation analysis, made on both H_t and returns (for comparison). We calculated the pairwise correlations (DJIA-FTSE, DJIA-N225 and FTSE-N225) for yearly, half-yearly and quarterly time horizons. The results are displayed in Figures 5(a)-(c).

4.2 Discussion of the results

As above recalled, the estimation of H_t serves two purposes: it is useful by itself to describe the state of each market at any time and is needed to analyze the pairwise cross-correlations between markets in terms of investors' perception of the information flow.
Concerning the first issue it is worthwhile pointing out that all the estimated H_t series:
(a) are centered around ½, what makes sense from a financial viewpoint, as discussed in Sections 1 and 2;
(b) are strongly mean reverting, what indicates that markets "continuously" correct the weights given to the past information;
(c) assume values exceptionally lower than ½ in presence of financial crises (October 1987 and Autumn 2008 are the most evident examples);
(d) are left-skewed (the skewness coefficients are: DJIA, –1.786; FTSE, –0.402; N225, –0.220), what indicates that the H_t's downturns are much more abrupt than the upwards movements (destroying memory is instantaneous, rebuilding it is a gradual process);

(e) tend to restore an equilibrium (i.e. to move towards ½) the more quickly the more destabilising is the shock experienced by the market.

Concerning the cross-correlation, the value-added by estimating it in terms of H_t rather than in terms of returns resides in the fact that this latent variable seizes how different markets perceive (one with respect to each other) the past information conditional to the arrival of new information. This inner knowledge permits to evaluate how accordingly markets discount the past. As a consequence, unlike domestic perturbations, a global shock is expected to reflect itself in a very high cross-correlation in terms of H_t.

With regard to the three analyzed series, we observe:

(a) while the correlations in terms of returns are generally below 0.5 for each scale, in terms of H_t we observe a more unstable and polarized behaviour in the shift from coarse-grained to fine-grained resolutions;

(b) the dynamics of the H_t cross-correlations appear significantly antipersistent when compared to the returns' ones. This is particularly evident for the DJIA-N225 and the FTSE-N225;

(c) as expected, the U.S. and the European indexes generally show higher correlations (both in terms of H_t and returns) with respect to the Japanese one;

(d) from the beginning of 2005 to the end of 2008 we register very high values in the H_t cross-correlations. In this four-year period markets seem to have processed information accordingly. A strong diversification among the three indexes arises from the beginning of 2009 (for the N225 we even observe a large negative correlation);

(e) the scale looks like acting differently for return and H_t correlations. For the former, values at fine-grained resolutions seem to interpolate coarse-grained data; for the latter resolution thickening increases the mean-reversion effect.

5. Conclusions and further developments

In this paper we have analyzed the pointwise regularity of three main stock indexes and studied the pairwise cross-correlations among them. The analysis of the H_t-based cross-correlations appears sensitive and timely and catches how the different markets react to the information flow, in terms of weight assigned to the past. The information provided by this analysis has evident practical consequences in terms of market-risk diversification.

Since in authors' knowledge this is the very first study that takes into consideration H_t as explaining latent variable, future works will be needed to check the results here provided and to extend the set of time series.

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Appendix A: List of figures

Figure 1. Bm, fBm and mBm

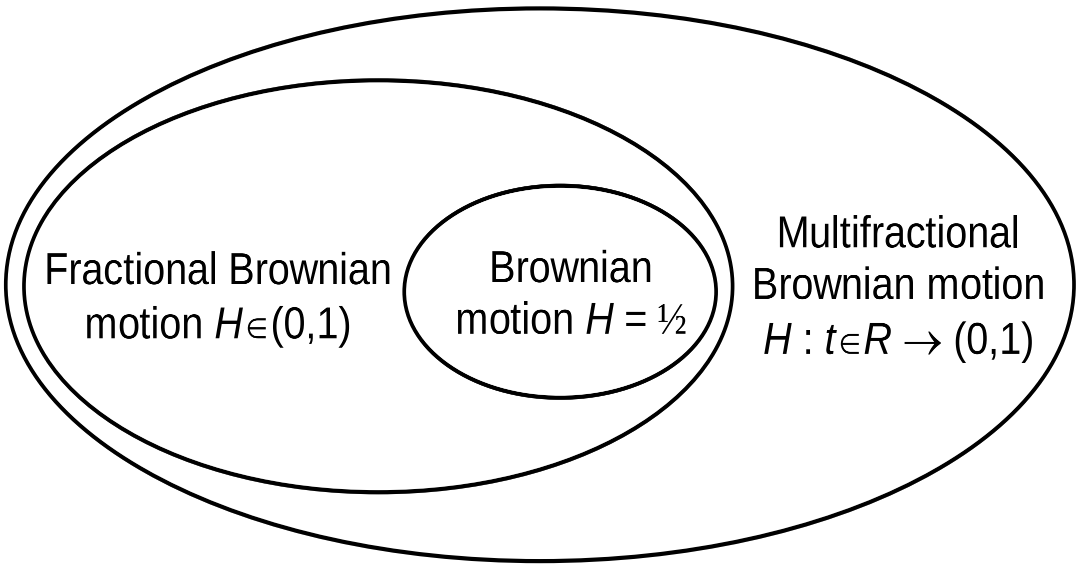


Figure 2. Autocovariance function of the increments of $B_H(t)$

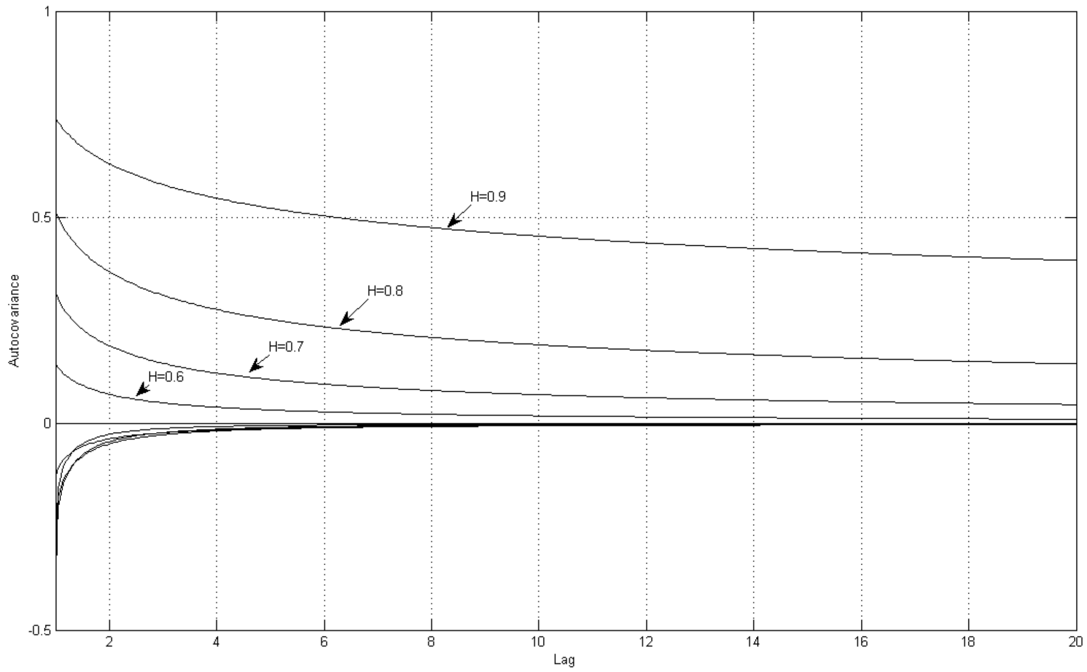
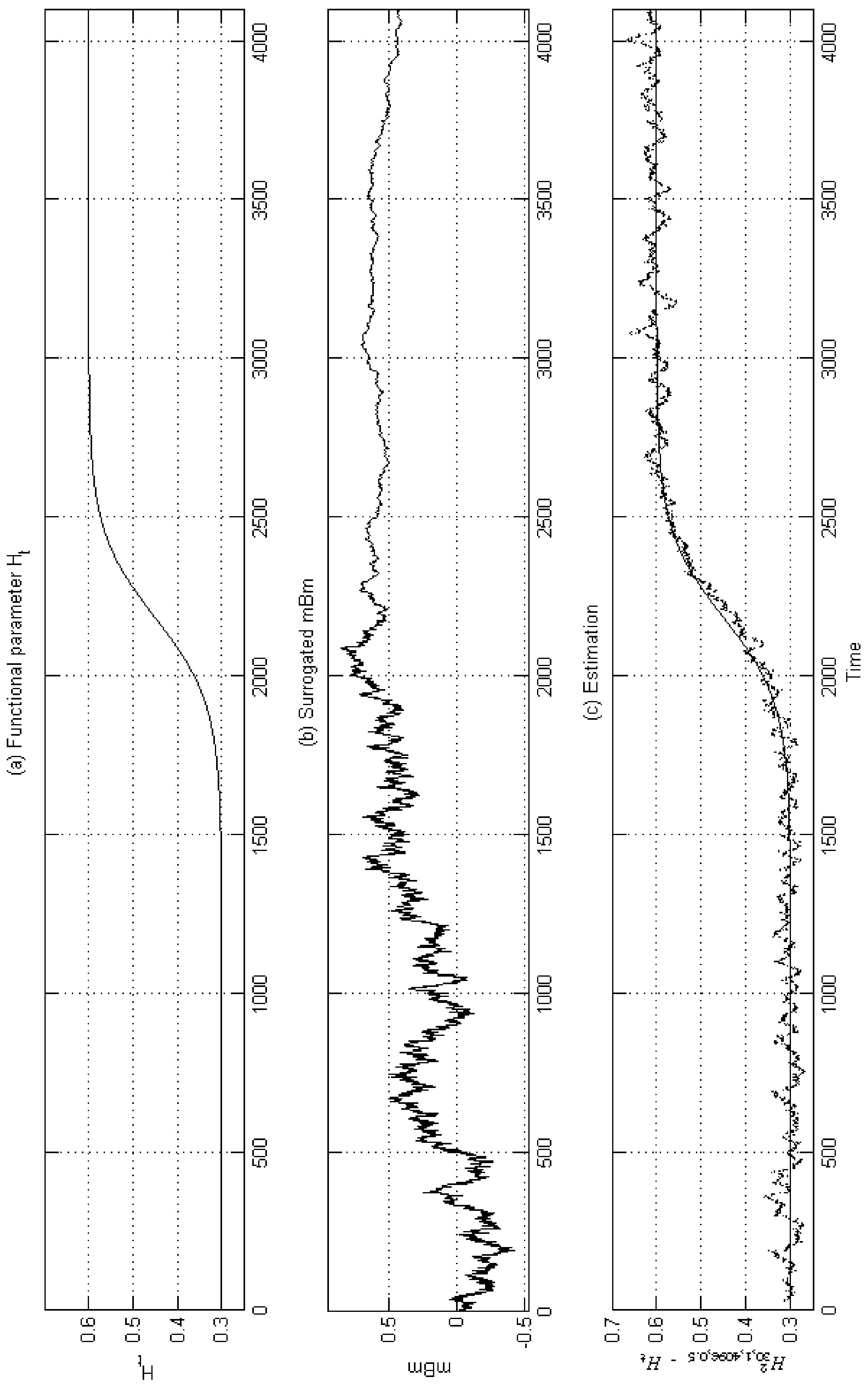
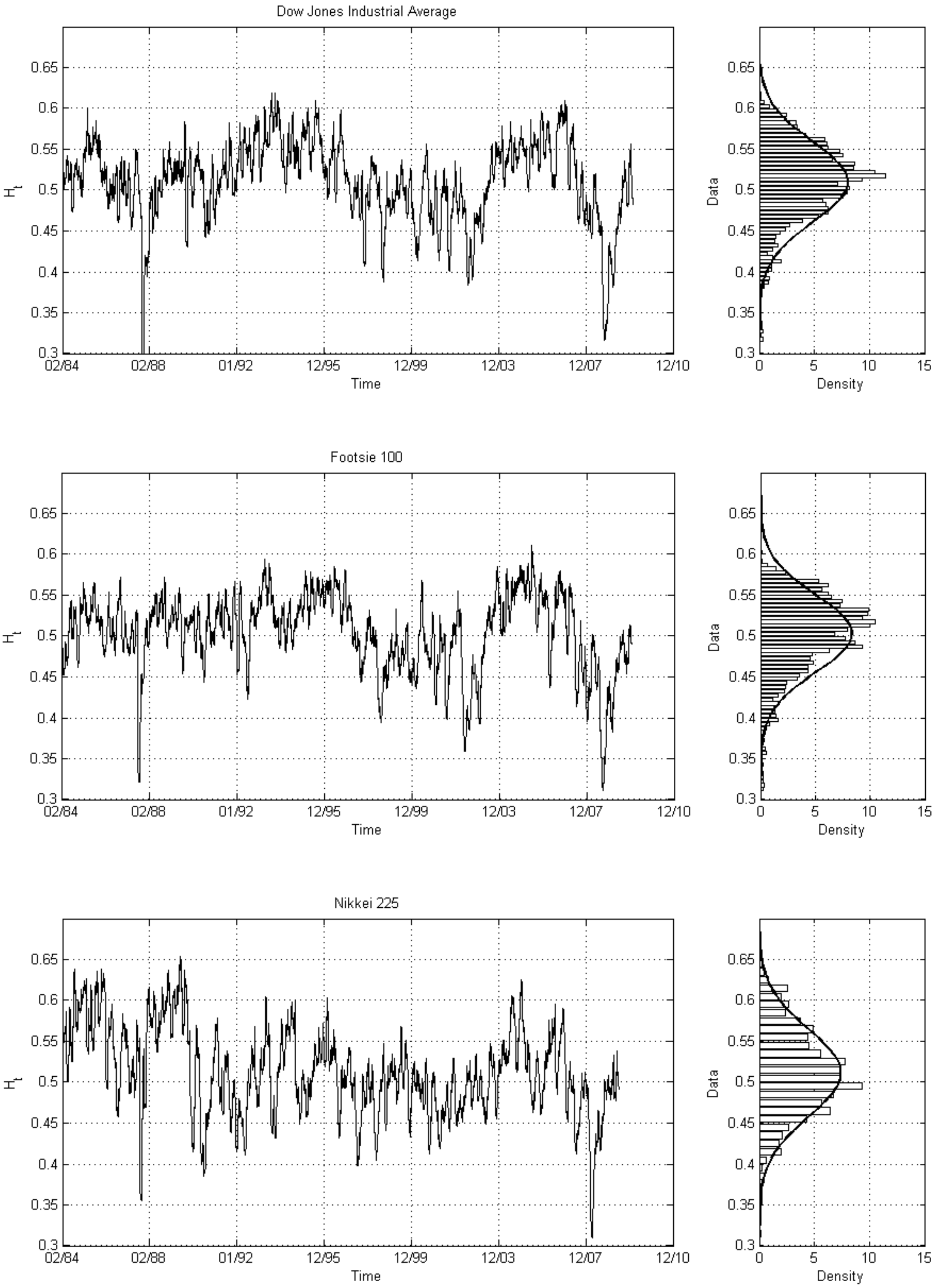


Figure 3. Estimation for surrogated mBm


(a) Functional parameter H_t
H_t
(b) Surrogated mBm
mBm
(c) Estimation
Time

Figures 4(a)-(c). H_t estimates



Figures 5(a)-(c). Pairwise correlations